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Abstract. We consider one-dimensional n-state quantum chains having as symmetry an Abelian group A of order n (generalised Ising chain). These Hamiltonians depend on $2(n-1)$ coupling constants. For special values of the coupling constants the symmetry of the quantum chains is larger than A corresponding to a (in general non-Abelian) group G with m elements ($m > n$). For each element of G we find explicitly a boundary condition which leaves the Hamiltonian translationally invariant. This problem is relevant in finite-size scaling studies at the critical point of a second-order phase transition.

1. Introduction

In this paper we consider one-dimensional n-state quantum chains which are defined on arbitrary Abelian groups A of finite order [1]. We want to investigate realisations of boundary conditions (BC) which are compatible with translational invariance (toroidal BC: TBC) of Hamiltonians belonging to n-state quantum chains. Considerations of different TBC are, for example, of interest in connection with investigations of finite-size spectra of one-dimensional n-state quantum chains in the infinite-volume limit at a critical point of a second-order phase transition. The finite-size spectra of these chains hint at irreducible and unitary representations (irreps) of two commuting Virasoro algebras with the same central charge c [2-9]. (The Hamiltonian of the quantum chain is supposed to be parity conserving.) An irrep of these Virasoro algebras is labelled by a scaling dimension x and a spin s of a corresponding scaling operator [2, 9, 10]. Operators of various spin s are realised by choosing appropriate TBC. Periodic BC in particular would enforce the appeerence of only integral spins. The full operator content of a given model can only be extracted by choosing several different TBC [6].

We are interested in finding an explicit representation for TBC for n-state quantum chains. To this end it is natural to exploit the relationship of these one-dimensional quantum chains (via transfer matrices) to two-dimensional Lagrangian systems [1]. To find TBC for two-dimensional Lagrange systems is a well known [11] and straightforward matter. The determination of the explicit form of TBC for n-state quantum chains also becomes simple through hints from the Lagrangian system. The result of the calculation is that TBC are introduced on the spin configuration space of n-state quantum chains by a similarity transformation given by a certain representation of the global symmetry group of the Hamiltonian considered. It turns out that this result is valid for all one-dimensional quantum chains with at most nearest-neighbour interaction. (It is valid also for those models which have no evident two-dimensional Lagrangian

counterpart.) On the other hand it is useful to introduce TBC to n-state quantum chains in a basis-independent manner. We show that this can be done for a certain class of symmetries of the Hamiltonians.

The paper is organised as follows. We briefly describe in the following section (§ 2) Lagrangians with TBC and a τ-continuum limit [1, 12] which results in Hamiltonians coinciding with special n-state quantum chains. This way one gets in an explicit manner TBC for a slightly restricted class of Hamiltonians which correspond to symmetries of the original Lagrangian. In § 3 we introduce general n-state quantum chains (which appear here as generalisations of those Hamiltonians obtained in § 2) and give a basis-invariant formulation of TBC as mentioned above. In § 4 we compute the possible TBC of a Z_8 model having as symmetry group $Z_2 \wr Z_2 \wr Z_2$. ('$\wr$' denotes here the wreath product of groups.) We present a table of the conjugacy classes of the $Z_2 \wr Z_2 \wr Z_2$ symmetry. (The finite-size spectra depend only on conjugacy classes.) Section 5 contains our conclusions.

The Z_8 model with Z_8 BC has already been investigated [8]. There it was shown by numerical methods that the model has a second-order phase transition at a certain temperature. Calculations based on TBC different from Z_8 will be presented elsewhere.

2. TBC for τ-continuum Hamiltonians

Two-dimensional Lagrange systems with short-range interaction are defined in generalisation to the Ising model on arbitrary finite Abelian groups A, so that the system is invariant under A translations. In character parametrisation the action S is [1]

$$S = \sum_{\mu=1}^{M} \sum_{\nu=1}^{N} aL(h_{\mu,\nu} - h_{\mu,\nu+1}) + bL(h_{\mu,\nu} - h_{\mu+1,\nu}) \tag{2.1}$$

$$L(h) = \sum_{r \neq 0} a_r \chi^r(h) \qquad h_{\mu,\nu}, h \in A.$$

M, N denotes the size of the two-dimensional lattice and 'a_r' are coupling constants. The parameters a and b describe an anisotropy of the couplings which will become relevant for the Hamiltonian limit we want to consider. $h_{\mu,\nu}$ and h are elements of the group A and $\chi^r(h)$ labels the character functions of the non-trivial irreps of A.

We choose periodic BC in the μ direction (the μ direction will become the time axis of the Hamilton version of the model (2.1)):

$$h_{M+1,\nu} := h_{1,\nu}. \tag{2.2}$$

In the ν direction we choose the boundary to be

$$h_{\mu,N+1} := \gamma(h_{\mu,1}) \tag{2.3}$$

where γ is a symmetry of L so that

$$L(h_1 - h_2) = L(\gamma(h_1) - \gamma(h_2)) \tag{2.4}$$

for arbitrary $h_1, h_2 \in A$. γ is supposed to map A bijectively onto itself.

We want to show that the Lagrangian (2.1) together with the BC (2.2) and (2.3) forms a translational invariant system. We introduce for this purpose an operator which generates translations. Let $K := \{k_{\mu,\nu}\}$, $k_{\mu,\nu} \in A$ be a configuration of spins. We

denote the configuration obtained by (twisted) translation of K in the μ direction (ν direction) by $T^\mu K$ ($T^\nu K$):

$$(T^\mu K)_{\mu,\nu} := \begin{cases} k_{\mu+1,\nu} & \text{if } \mu \neq M \\ k_{1,\nu} & \text{if } \mu = M \end{cases} \tag{2.5}$$

$$(T^\nu K)_{\mu,\nu} := \begin{cases} k_{\mu,\nu+1} & \text{if } \nu \neq N \\ \gamma(k_{\mu,1}) & \text{if } \nu = N. \end{cases} \tag{2.6}$$

One immediately verifies that the action (2.1) with the BC (2.2) and (2.3) is invariant under translations. This remains true if the model is reformulated in the transfer matrix language in which the μ direction is chosen to be the time axis. Translational invariance in the μ direction is, in the transfer matrix formalism, expressed by the fact that the system is described by a single transfer matrix, while translational invariance in the ν direction is a consequence of the fact that the transfer matrix commutes with an operator T^γ being the counterpart of the operator (2.6) which generates in the Lagrangian formalism translations in the ν direction. As our considerations are completely independent from the anisotropy introduced in the action (2.1) there is no reason to doubt that the transfer matrix remains translationally invariant even in the extreme anisotropic limit (τ continuum limit) which defines special n-state quantum chains. It should therefore be possible to recognise the general form of TBC, at least for the n-state quantum chains corresponding to the transfer matrices which we consider here.

To define the transfer matrix in the τ-continuum limit we have to consider the partition function Z corresponding to the action (2.1)

$$Z = \sum_{\text{config}} \exp\{-S\} \tag{2.7}$$

in the limit $b \to \infty$ and $a \to 0$ where the limit is taken so that $\lambda = a \cdot \exp\{b \cdot k_\theta\}$ remains at a fixed value. Here k_θ denotes the minimum of $L(h)$ over A and θ denotes the subset of A where $L(h) = k_\theta$. Since it is well known how to perform this τ continuum limit (see, e.g., [1, 12]), we restrict ourselves to summarising the results. (For details see [1].) We get the following Hamiltonian H_γ:

$$H_\gamma = -\sum_{\nu=1}^{N} \sum_{h \in \theta} \sigma_\nu^h - \lambda \left(\sum_{\nu=1}^{N-1} \sum_{r \neq 0} a_r \Gamma_\nu^r \Gamma_{\nu+1}^{r^+} + \sum_{r \neq 0} a_r \Gamma_N^r \Gamma_1^{r^+}(\gamma) \right) \tag{2.8}$$

where

$$\sigma_\nu^h = 1 \otimes \ldots \otimes \underbrace{1 \otimes \sigma^h \otimes 1}_{\nu\text{th position}} \otimes \ldots \otimes 1$$

$$\Gamma_\nu^r = 1 \otimes \ldots \otimes 1 \underbrace{\otimes \Gamma^r \otimes}_{\nu\text{th position}} 1 \otimes \ldots \otimes 1$$

$$\Gamma_1^r(\gamma) = \Gamma^r(\gamma) \otimes 1 \otimes \ldots \otimes 1$$

and σ^h, Γ^r and $\Gamma^r(\gamma)$ are operators defined on the n-dimensional (n is the order of the finite Abelian group A) space of functions defined on A with range in the complex numbers C. A natural base in this space is given by $B = \{e_g : A \to C \mid e_g(h) = \delta(g - h)g, h \in A\}$. The effect of the operators σ^h, Γ^r and $\Gamma^r(\gamma)$ on the base B is

$$\sigma^h e_g = e_{g-h} \qquad \Gamma^r e_g = \chi^r(g) e_g \qquad g, h \in A; e_g \in B \tag{2.9}$$

where $\chi^r(g)$ label the characters of the irreps of A.

Finally the boundary term is given by

$$\Gamma'(\gamma) = \chi^r(\gamma(g))e_g$$
$$= U^{\dagger}(\gamma)\Gamma'U(\gamma)e_g \qquad (2.10)$$

$$U(\gamma)e_g = e_{\gamma(g)} \qquad (2.11)$$

where $U(\gamma)$ is nothing but a permutation of the base B.

The translation operator T^{ν} (2.5) in the ν direction belonging to the BC (2.3) becomes, in the transfer matrix language,

$$T^{\gamma} = U_1(\gamma)T. \qquad (2.12)$$

T is the translation operator of periodic BC and explicitly given by

$$T(e_{g1} \otimes \ldots \otimes e_{gN}) = e_{gN} \otimes e_{g1} \otimes \ldots \otimes e_{gN-1} \qquad (2.13)$$

and $U_1(\gamma)$ is defined by

$$U_1(\gamma) = U(\gamma) \otimes 1 \otimes \ldots \otimes 1. \qquad (2.14)$$

One easily verifies that T^{γ} and H_{γ} in fact commute with each other. One has to use for this purpose the identities

$$0 = \left[\sum_{h \in \theta} \sigma_N^h, \tilde{U}(\gamma) \right] = \left[\sum_{r \neq 0} a_r \Gamma_1^r \Gamma_2^{r^{\dagger}}, \tilde{U}(\gamma) \right]$$
$$\tilde{U}(\gamma) = \bigotimes_{j=1}^{N} U(\gamma) \qquad (2.15)$$

which are consequences of the fact that γ is a symmetry of the Lagrangian L.

Equations (2.10)-(2.12) together with (2.15) solve the problem of TBC for the Hamiltonians given by (2.8). More general Hamiltonians will be considered in the following section.

3. General n-state quantum chains

Using the same notation as in § 2 we define general n-state quantum chains in agreement with [1] by

$$H = -\sum_{j=1}^{N} \left(\sum_{h \in A \setminus \{0\}} b_h \sigma_j^h + \lambda \sum_{r \neq 0} a_r \Gamma_j^r \Gamma_{j+1}^{r^{\dagger}} \right). \qquad (3.1)$$

The generalisation in comparison with (2.8) consists in the introduction of arbitrary coupling constants $\{b_h\}$ for the one-site terms. We start with periodic BC:

$$\Gamma_{N+1}^r = \Gamma_1^r \qquad (3.2)$$

which obviously belong to TBC. The symmetry of the Lagrangian system played, as we saw in § 2, a crucial role for the definition of TBC. We investigate therefore the invariance group of the system (3.1).

The system is, by construction, invariant under A translations:

$$\tilde{U}(h)\sigma_j^h \tilde{U}(h) = \sigma_j \qquad \tilde{U}^{\dagger}(h)\Gamma_j^r \Gamma_{j+1}^{r^{\dagger}} \tilde{U}(h) = \Gamma_j^r \Gamma_{j+1}^{r^{\dagger}}$$

$$\tilde{U}(h) = \bigotimes_{j=1}^{N} U(h) \qquad (3.3)$$

$$U(h)e_g = e_{g+h} \qquad U^{\dagger}(h)e_g = e_{g-h} \qquad g, h \in A; \ e_g \in B.$$

It might happen that the symmetry of the Hamiltonian is greater than the group A. In order to determine further symmetries of the Hamiltonian (3.1) we consider two particular Lagrange systems (3.4) [1] living on the same Abelian group A as the Hamiltonian chain. These Lagrange systems are given finally by the coupling constants $\{a_r\}_{r\neq 0}$ and $\{b_h\}_{h\in A}$ defining the quantum chain (3.1). The Lagrange densities L_O, L_C of these Lagrangian systems are

$$L_O = \sum_{h\in A\setminus\{0\}} b_h\delta(g-h) \qquad L_C = \sum_{r\neq 0} a_r\chi^r(g) \tag{3.4}$$

where L_O is given in the so-called orbit parametrisation and obviously corresponds to that part of the Hamiltonian (3.1) which does not depend on λ, while L_C is given in the character representation and corresponds to the term of the Hamiltonian (3.1) which is linear in λ. We consider these Lagrange systems since there is a relationship between their invariance groups and the invariance group of the Hamiltonian chain (3.1). This relationship is established by a mapping which associates with each permutation γ acting on the group A an orthogonal transformation $U(\gamma)$ defined on the n-dimensional spin space of the Hamiltonian chain (3.1). If γ is any permutation of the group elements of A we define $U(\gamma)$ by

$$U(\gamma)e_g = e_{\gamma(g)}. \tag{3.5}$$

If we choose γ to be an element of the symmetry group G_O (G_C) of the Lagrangian L_O (L_C), $\tilde{U}(\gamma) = \bigotimes_{j=1}^{N} U(\gamma)$ commutes with the part of the Hamiltonian to which the Lagrangian L_O (L_C) is related. A symmetry of the entire Hamiltonian has to commute with both parts of the Hamiltonian. This will automatically be the case if we choose the symmetry γ to be in the intersection G_{OC} of the symmetry groups G_O, G_C. So we have

$$\left[\sum_{h\in A\setminus\{0\}} b_h\sigma_N^h, \tilde{U}(\gamma)\right] = \left[\sum_{r\neq 0} a_r\Gamma_1^r\Gamma_2^{r^\dagger}, \tilde{U}(\gamma)\right] = 0 \qquad \gamma\in G_{OC}. \tag{3.6}$$

One easily recognises in (3.6) the generalisations of the equation (2.15). We conclude that we can write TBC for each element of G_{OC} belonging to the invariance group of H, simply by using the expressions (2.10)–(2.12) for $\gamma\in G_{OC}$.

$$\Gamma_{N+1}^r = U_1^\dagger(\gamma)\Gamma_1^r U_1(\gamma)$$
$$T^\gamma = U_1(\gamma)T. \tag{3.7}$$

The form of these TBC has the drawback that they are defined in respect to a special base, and they are rather remote from other more familiar TBC. Therefore our next aim is to show that both objections can be removed for those symmetries of the Hamiltonian chain which belong to G_{OC}. We will show that TBC can be given for these symmetries in the common basis invariant form as linear combinations of the operators Γ_1^r [3, 7]:

$$\Gamma_{N+1}^r = \sum_{r'\neq 0} D^{r,r'}\Gamma_1^{r'}. \tag{3.8}$$

One has to use the following orthogonality relations to this end, valid for Abelian and finite groups A of order n [13]:

$$\sum_{h\in A} \chi^r(h)\chi^{r'^*}(h) = n\delta_{r,r'} \tag{3.9}$$

$$\sum_r \chi^r(g)\chi^{r^*}(h) = n\delta(h-g) \qquad h, g\in A \tag{3.10}$$

where $\chi^r(h)$ are the characters of the irreps of A, as defined before. These relations yield:

$$U^\dagger(\gamma)\Gamma^r U(\gamma) = \chi^r(\gamma(g))e_g$$

$$= \sum_{h \in A} \chi^r(\gamma(h))\delta(h-g)e_g$$

$$= \sum_{r'} \frac{1}{n} \sum_{h \in A} \chi^r(\gamma(h))\chi^{r'^*}(h)\chi^{r'}(g)e_g. \qquad (3.11)$$

We may rewrite (3.11) as

$$U^\dagger(\gamma)\Gamma^r U(\gamma) = \sum_{r' \neq 0} D^{r,r'}(\gamma)\Gamma^{r'} \qquad (3.12)$$

where

$$D^{r,r'}(\gamma) = \frac{1}{n} \sum_{h \in A} \chi^r(\gamma(h))\chi^{r'^*}(h). \qquad (3.13)$$

Equation (3.13) may be regarded as the main result of this paper. We have managed to make contact to the more familiar basis-invariant form of the TBC. It can be shown in general with help of the orthogonality relations (3.9) and (3.10) that the set of matrices $D(\gamma)$, with γ being a symmetry element of the n-state quantum chain form a unitary, not necessary irreducible, but faithful representation of the invariance group of H. This property of the boundary conditions was previously observed in special cases, e.g. in the four-state Potts model [7] or in the XXZ model [14]. This latter model is not contained in the system we considered.

4. An application

The model which we want to consider here is given by equation (3.1) specialised to the group Z_8. We demand in addition that the coupling constants satisfy certain relations:

$$a_r = b_r$$

$$a_2 = a_4 = a_6 \qquad (4.1)$$

$$a_1 = a_3 = a_5 = a_7.$$

This Hermitian model has been investigated in [8] where it was stated that it is conformally invariant for the temperature λ taking the value one. The Virasoro charge c was found to be $1.3\ldots$ [8].

The symmetry of this model is determined by the Lagrangian L_C as introduced in § 3. This can be seen by rewriting this Lagrangian in orbit parametrisation. The comparison of the two Lagrange densities L_C and L_O leads to the conclusion that the orbits of the Lagrangian L_C are contained in the orbits of L_O so that each symmetry of L_C is also a symmetry of L_O. This confirms our statement. The symmetry group is $Z_2 \wr Z_2 \wr Z_2$, where '$\wr$' denotes the wreath product of groups [1, 15]. The group is

generated by the following four permutations acting on the group Z_8:

$$g_1 = (01234567) \qquad g_2 = (17)(26)(35)$$
$$g_3 = (1357) \qquad g_4 = (37). \tag{4.2}$$

g_1 generates Z_8 translations and g_2 charge conjugation and both generate the dihedral group D_8 for which TBC are known. Each element g of $Z_2 \wr Z_2 \wr Z_2$ can be represented by

$$g = g_1^l g_2^m g_3^n g_4^o \qquad l, m, n, o \text{ integers.} \tag{4.3}$$

The order of this group is 128. A careful investigation of the group $Z_2 \wr Z_2 \wr Z_2$ shows, that it contains two $Z_2 \otimes Z_2 \otimes Z_2$ acting transitively on Z_8. From the existence of such a subgroup $(Z_2 \otimes Z_2 \otimes Z_2)$ of the invariance group of the Hamiltonian follows that this Z_8 model is isomorphic to a model defined on the Abelian group $Z_2 \otimes Z_2 \otimes Z_2$ [16]. This isomorphism may be realised, for example, by the following map:

$$
\begin{array}{ccc}
Z_8 & & Z_2 \otimes Z_2 \otimes Z_2 \\
0 & \leftrightarrow & 000 \\
1 & \leftrightarrow & 100 \\
2 & \leftrightarrow & 001 \\
3 & \leftrightarrow & 010 \\
4 & \leftrightarrow & 111 \\
5 & \leftrightarrow & 011 \\
6 & \leftrightarrow & 110 \\
7 & \leftrightarrow & 101.
\end{array}
\tag{4.4}
$$

The classification of models defined on $Z_2 \otimes Z_2 \otimes Z_2$ has already been done in [1]. A comparison with the table given in [1] yields the $Z_2 \wr Z_2 \wr Z_2$ invariance of the model.

The 20 conjugacy classes of $Z_2 \wr Z_2 \wr Z_2$ are given in table 1. From (4.3) and the homomorphism property of the representation D of the invariance group $Z_2 \wr Z_2 \wr Z_2$ stated in § 3 one infers that the 128 TBC may be represented in terms of four basic matrices:

$$D(g) = D(g_1^l g_2^m g_3^n g_4^o) = D(g_1)^l D(g_2)^m D(g_3)^n D(g_4)^o. \tag{4.5}$$

As the spectrum of the Hamiltonian system (3.1) only depends on the conjugacy class which the group element determining the TBC of the quantum chain belongs to, one has to consider only one arbitrary TBC for each conjugacy class. Consequently only 20 TBC are relevant for computations. Suitable representatives of the conjugacy classes may be selected out of table 1. Using that the characters of Z_8 are given by

$$\chi^r(h) = \omega^{rh} \qquad \omega = \exp\left\{ i \cdot \frac{2\pi}{8} \right\} = \frac{1+i}{\sqrt{2}} \qquad r, h \in \{0, \dots, 7\} \tag{4.6}$$

Table 1. Conjugacy classes of Z_2 Z_2 Z_2

Conjugacy class	Order of elements	Number of elements
1	1	1
g_4; $g_3^2 g_4$; $g_2 g_3^3 g_4$; $g_1^4 g_2 g_3 g_4$	2	4
g_3; g_3^3; $g_1^2 g_3^3$; $g_1^6 g_3$	4	4
$g_3 g_4$; $g_3^3 g_4$; $g_1^2 g_2 g_4$; $g_1^6 g_2 g_3^2 g_4$	2	4
g_3^2; $g_1^4 g_3^2$	2	2
g_2; $g_2 g_3^2$; $g_1^6 g_2$; $g_1^6 g_2 g_3^2$; $g_1^2 g_2$; $g_1^2 g_2 g_3^2$; $g_1^4 g_2$; $g_1^4 g_2 g_3^2$	2	8
$g_2 g_4$; $g_2 g_3^2 g_4$; $g_1^6 g_3^3 g_4$; $g_1^6 g_3 g_4$; $g_1^2 g_3^3 g_4$; $g_1^2 g_3 g_4$; $g_1^4 g_2 g_3^2 g_4$; $g_1^4 g_2 g_4$	4	8
$g_2 g_3$; $g_2 g_3^3$; $g_1^4 g_2 g_3^3$; $g_1^4 g_2 g_3$	2	4
$g_2 g_3 g_4$; $g_1^4 g_4$; $g_1^4 g_3^2 g_4$; $g_1^2 g_2 g_3^3 g_4$	2	4
g_1; $g_1 g_2 g_3^3$; $g_1^7 g_3^2$; $g_1^3 g_2 g_3^3$; g_1^5; $g_1^5 g_2 g_3^3$; $g_1^3 g_3^2$; $g_1^7 g_2 g_3^3$; g_1^7; $g_1^3 g_2 g_3$; $g_1^5 g_3^2$; $g_1^5 g_2 g_3$; g_1^3; $g_1^7 g_2 g_3$; $g_1 g_3^2$; $g_1 g_2 g_3$	8	16
$g_1 g_4$; $g_1 g_2 g_3^3 g_4$; $g_1^7 g_4$; $g_1^3 g_2 g_3 g_4$; $g_1^5 g_4$; $g_1^5 g_2 g_3^3 g_4$; $g_1^3 g_4$; $g_1^7 g_2 g_3 g_4$; $g_1^7 g_3^2 g_4$; $g_1^3 g_2 g_3^3 g_4$; $g_1^5 g_3^2 g_4$; $g_1^5 g_2 g_3 g_4$; $g_1^3 g_3^2 g_4$; $g_1^7 g_2 g_3^3 g_4$; $g_1 g_3^2 g_4$; $g_1 g_2 g_3 g_4$	4	16
$g_1 g_3$; $g_1 g_2 g_3^2$; $g_1^7 g_3^3$; $g_1^3 g_2 g_3^2$; $g_1^5 g_3$; $g_1^5 g_2 g_3^2$; $g_1^3 g_3^3$; $g_1^7 g_2 g_3^2$	4	8
$g_1 g_3 g_4$; $g_1 g_2 g_3^2 g_4$; $g_1^7 g_3 g_4$; $g_1^3 g_2 g_4$; $g_1^5 g_3 g_4$; $g_1^5 g_2 g_3^2 g_4$; $g_1^3 g_3 g_4$; $g_1^7 g_2 g_4$; $g_1 g_3^3 g_4$; $g_1 g_2 g_4$; $g_1^7 g_3^3 g_4$; $g_1^3 g_2 g_3^2 g_4$; $g_1^5 g_3^3 g_4$; $g_1^5 g_2 g_4$; $g_1^3 g_3^3 g_4$; $g_1^7 g_2 g_3^2 g_4$	4	16
$g_1 g_3^3$; $g_1 g_2$; $g_1^7 g_3^3$; $g_1^3 g_2$; $g_1^5 g_3^3$; $g_1^5 g_2$; $g_1^3 g_3$; $g_1^7 g_2$	2	8
g_1^2; $g_1^2 g_3^2$; g_1^6; $g_1^6 g_3^2$	4	4
$g_1^2 g_4$; $g_1^2 g_3^2 g_4$; $g_1^6 g_3^2 g_4$; $g_1^6 g_4$; $g_1^2 g_2 g_3^3 g_4$; $g_1^2 g_2 g_3 g_4$; $g_1^6 g_2 g_3^3 g_4$; $g_1^6 g_2 g_3 g_4$	4	8
$g_1^2 g_3$; $g_1^6 g_3^3$; $g_1^4 g_3^3$; $g_1^4 g_3$	4	4
$g_1^2 g_2 g_3$; $g_1^2 g_2 g_3^3$; $g_1^6 g_2 g_3^3$; $g_1^6 g_2 g_3$	2	4
$g_1^2 g_2 g_3^2 g_3 g_4$; $g_1^6 g_2 g_4$; $g_1^4 g_3 g_4$; $g_1^3 g_3^3 g_4$	2	4
g_1^4	2	1

one yields the following matrices:

$$A(g_1) = \begin{pmatrix} \omega & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega^5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega^6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \omega^7 \end{pmatrix}$$

$$A(g_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A(g_3) = \begin{pmatrix} \tfrac{1}{2}(1+i) & 0 & 0 & 0 & \tfrac{1}{2}(1-i) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \tfrac{1}{2}(1-i) & 0 & 0 & 0 & \tfrac{1}{2}(1+i) \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \tfrac{1}{2}(1-i) & 0 & 0 & 0 & \tfrac{1}{2}(1+i) & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tfrac{1}{2}(1+i) & 0 & 0 & 0 & \tfrac{1}{2}(1-i) \end{pmatrix}$$

$$A(g_4) = \begin{pmatrix} \tfrac{1}{2} & 0 & -\tfrac{1}{2}i & 0 & \tfrac{1}{2} & 0 & \tfrac{1}{2}i \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \tfrac{1}{2}i & 0 & \tfrac{1}{2} & 0 & -\tfrac{1}{2}i & 0 & \tfrac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \tfrac{1}{2} & 0 & \tfrac{1}{2}i & 0 & \tfrac{1}{2} & 0 & -\tfrac{1}{2}i \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -\tfrac{1}{2}i & 0 & \tfrac{1}{2} & 0 & \tfrac{1}{2}i & 0 & \tfrac{1}{2} \end{pmatrix}. \tag{4.7}$$

One recognises the common TBC generated by $D(g_1)$, $D(g_2)$ corresponding to D_8 [2, 3].

5. Conclusion

We presented a detailed investigation of TBC for n-state quantum chains defined on finite groups. Starting from the classical two-dimensional Lagrange formalism in § 2, we discussed the introduction of TBC by performing an anisotropic continuum time limit. We derived expressions for TBC and their corresponding translation operator in the Hamiltonian picture. These expressions could easily be represented by permutations $U(\gamma)$ of the base $B = \{e_g \mid e_g(h) = \delta(g-h)g, h \in A\}$ of the n-dimensional spin space belonging to the Abelian group A where the Hamiltonian H is defined on:

$$\Gamma'_{N+1} = U_1^{\dagger}(\gamma)\Gamma'_1 U_1(\gamma) \qquad T^{\gamma} = U_1(\gamma)T \tag{5.1}$$

where T is the translation operator of periodic BC. $U_1(\gamma)$ acts on the first site of the lattice through $U(\gamma)$. γ is a symmetry of the original Lagrange system. The result was extended to more general Hamiltonians in § 3. Equation (5.1) is valid even for the general n-state quantum chain as defined in § 3 (3.1).

Although the TBC (5.1) are rather natural, because they can be interpreted geometrically as having closed the chain up to a rotation in the spin space, (5.1) seems not to coincide with convenient TBC. We discussed this problem in § 3 and derived a basis-invariant formulation of (5.1) for symmetries of n-state quantum chains which are associated with permutations $U(\gamma)$ of the base B. This formulation agreed, for special symmetries $U(D_m)$, with the known TBC of Z_m models with D_m symmetry. The possibility to define TBC basis-independent reflects the fact that (5.1) can be written as a linear combination of the matrices Γ'_1. So we have the following explicit formula

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for TBC:

$$\Gamma^r_{N+1} = U^\dagger(\gamma)\Gamma^r U(\gamma) = \sum_{r'\neq 0} D^{r,r'}(\gamma)\Gamma^{r'} \tag{5.2}$$

where $D(\gamma)$ is given here by

$$D^{r,r'}(\gamma) = \frac{1}{n}\sum_{h\in A} \chi^r(\gamma(h))\chi^{r'*}(h) \tag{5.3}$$

where $\chi^r(g)$ are the n characters belonging to the irreps of the Abelian group A of order n. The fact that the TBC (5.1) can be given basis invariant by (5.2) is a consequence of the relationship of the considered symmetry group of the Hamiltonian to that of Lagrangian systems.

The formula (5.3) represents the main result of this paper. It reflects in a simple and basis-invariant form the connection between the global symmetries of the Hamiltonian and possible TBC.

In § 4 we gave an application to a Z_8 model with $Z_2 \wr Z_2 \wr Z_2$ symmetry.

Acknowledgments

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